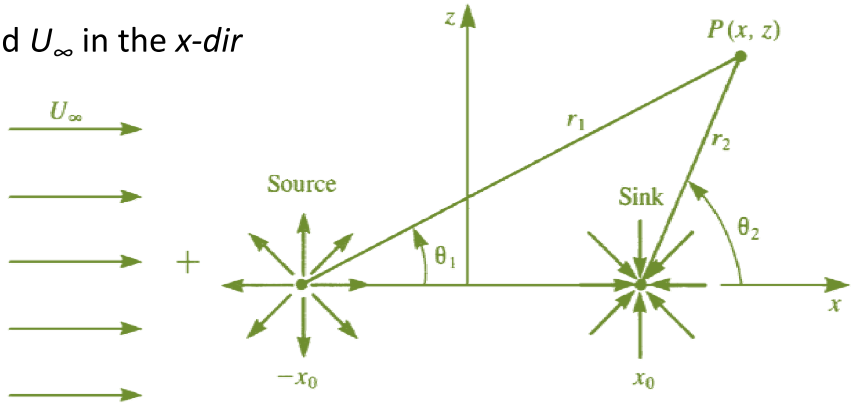
3.10 Superposition of Sources and Free Stream: Rankine's Oval

2D flow
- Source σ at $x = -x_0$
- Sink $-\sigma$ at $x = +x_0$
- Uniform flow with speed U_∞ in the *x-dir*

$$r_1 = [(x + x_0)^2 + z^2]^{1/2}$$
$$r_2 = [(x - x_0)^2 + z^2]^{1/2}$$
$$\theta_1 = \tan^{-1}[z/(x + x_0)]$$
$$\theta_2 = \tan^{-1}[z/(x - x_0)]$$



$$\Phi(x, z) = U_\infty x + \frac{\sigma}{2\pi}\ln(r_1) - \frac{\sigma}{2\pi}\ln(r_2) \quad (3.87)$$ *Velocity Potential*

$$\Psi(x, z) = U_\infty z + \frac{\sigma}{2\pi}\theta_1 - \frac{\sigma}{2\pi}\theta_2 \quad (3.88)$$ *Stream Function*

$$\Phi(x, z) = U_\infty x + \frac{\sigma}{2\pi}\ln\sqrt{(x + x_0)^2 + z^2} - \frac{\sigma}{2\pi}\ln\sqrt{(x - x_0)^2 + z^2} \quad (3.87a)$$

$$\Psi(x, z) = U_\infty z + \frac{\sigma}{2\pi}\tan^{-1}\frac{z}{x + x_0} - \frac{\sigma}{2\pi}\tan^{-1}\frac{z}{x - x_0} \quad (3.88a)$$

3.10 Superposition of Sources and Free Stream: Rankine's Oval

The velocity field due to this potential

$$u = \frac{\partial \Phi}{\partial x} = U_\infty + \frac{\sigma}{2\pi}\frac{x + x_0}{(x + x_0)^2 + z^2} - \frac{\sigma}{2\pi}\frac{x - x_0}{(x - x_0)^2 + z^2} \quad (3.89)$$

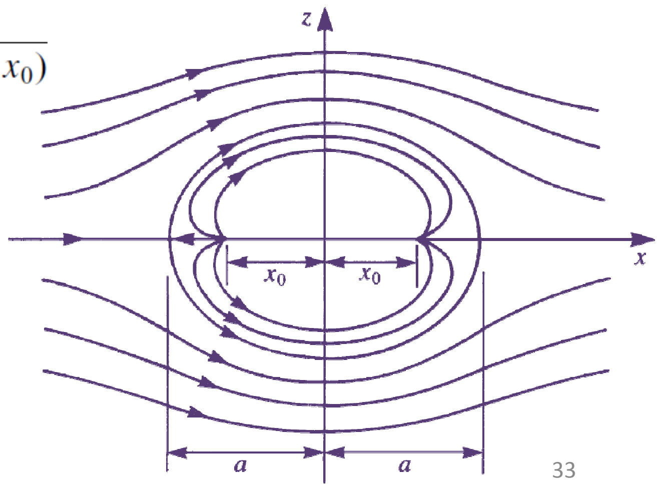
$$w = \frac{\partial \Phi}{\partial z} = \frac{\sigma}{2\pi}\frac{z}{(x + x_0)^2 + z^2} - \frac{\sigma}{2\pi}\frac{z}{(x - x_0)^2 + z^2} \quad (3.90)$$

The stagnation Points at $x = \pm a$ $u = 0$

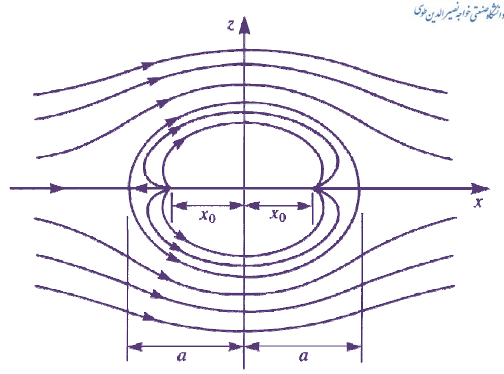
$$u(\pm a, 0) = U_\infty + \frac{\sigma}{2\pi}\frac{1}{(\pm a + x_0)} - \frac{\sigma}{2\pi}\frac{1}{(\pm a - x_0)}$$

$$= U_\infty - \frac{\sigma}{\pi}\frac{x_0}{(a^2 - x_0^2)} = 0$$

$$a = \sqrt{\frac{\sigma x_0}{\pi U_\infty} + x_0^2}$$



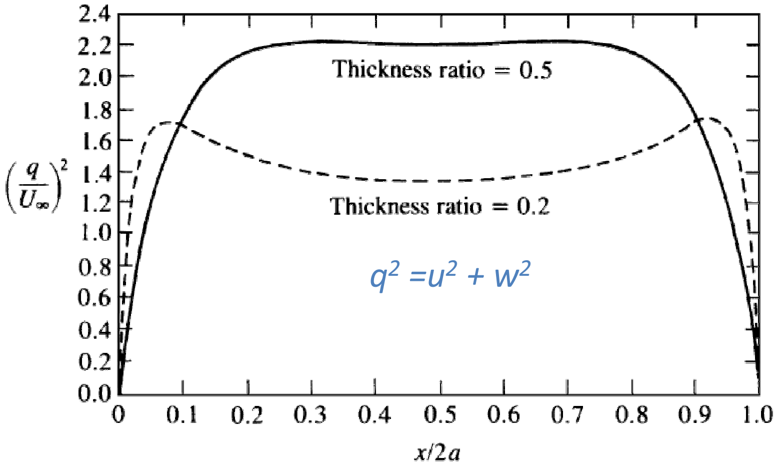
3.10 Superposition of Sources and Free Stream: Rankine's Oval



The stagnation streamline

$$\begin{cases} \theta_1 = \theta_2 = \pi \ \& \ z = 0 \\ \theta_1 = \theta_2 = 0 \ \& \ z = 0 \end{cases} \longrightarrow \text{Eq.(3.88)} \longrightarrow \Psi = 0$$

$$\Psi(x, z) = U_\infty z + \frac{\sigma}{2\pi} \tan^{-1} \frac{z}{x + x_0} - \frac{\sigma}{2\pi} \tan^{-1} \frac{z}{x - x_0} = 0 \qquad (3.92)$$



velocity distribution
Rankine's oval

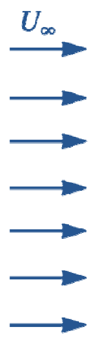
3.11 Flow around a Cylinder

Superposition of Doublet and Free Stream

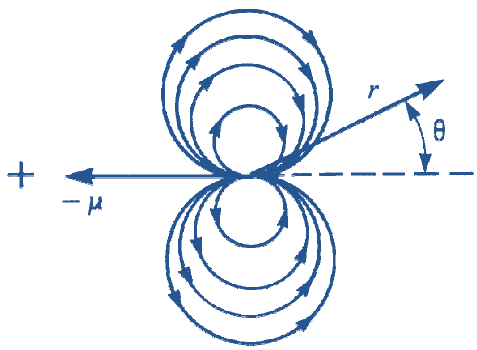
$$\Phi = U_\infty r \cos\theta + \frac{\mu}{2\pi} \frac{\cos\theta}{r} \qquad (3.93)$$

$$q_r = \frac{\partial \Phi}{\partial r} = \left(U_\infty - \frac{\mu}{2\pi r^2}\right) \cos\theta \qquad (3.94)$$

$$q_\theta = \frac{1}{r} \frac{\partial \Phi}{\partial \theta} = -\left(U_\infty + \frac{\mu}{2\pi r^2}\right) \sin\theta \qquad (3.95)$$



Streamlines for a uniform flow



Streamlines for a doublet $[\mu = (-\mu, 0)]$

r = R as the radius of the circle → strength of the doublet $\mu = U_\infty 2\pi R^2$

$$\Phi = U_\infty \cos\theta \left(r + \frac{R^2}{r}\right) \qquad (3.97)$$

$$q_r = U_\infty \cos\theta \left(1 - \frac{R^2}{r^2}\right) \qquad (3.98)$$

$$q_\theta = -U_\infty \sin\theta \left(1 + \frac{R^2}{r^2}\right) \qquad (3.99)$$

3.11 Flow around a Cylinder (Continue)

For 2D case

$$\Psi = U_\infty r \, \sin\theta - \frac{\mu}{2\pi}\frac{\sin\theta}{r} \qquad (3.100)$$

The stagnation points on the circle

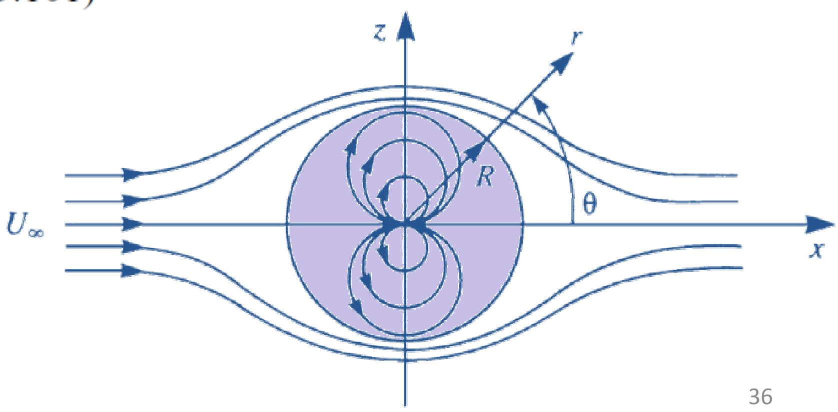
$q_\theta = 0$ in Eq. (3.99) → $\theta = 0$ and $\theta = \pi$ → $\Psi = 0$

The streamlines of the flow around the cylinder with radius *R*

$$\Psi = U_\infty \sin\theta \left(r - \frac{R^2}{r} \right) \qquad (3.101)$$

Eq. (3.100)

$$\mu = U_\infty 2\pi R^2$$



3.11 Flow around a Cylinder (Continue)

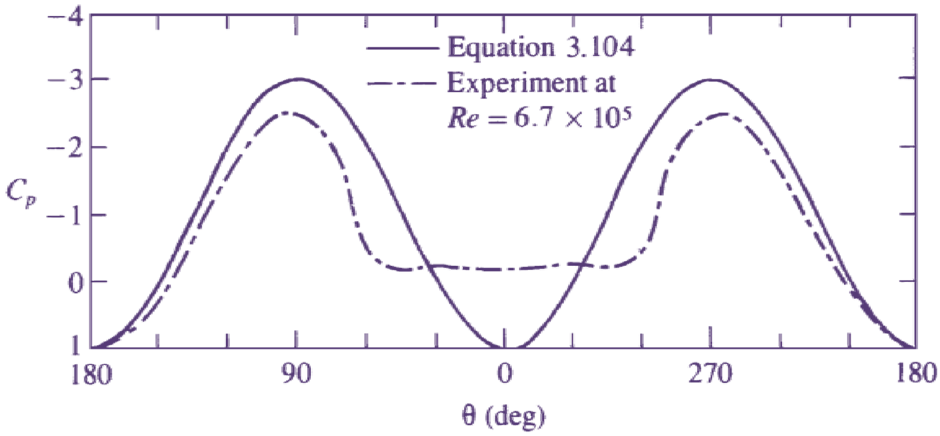
The pressure distribution over the cylinder at *r* = *R*

$$q_r = 0, \qquad q_\theta = -2U_\infty \sin\theta$$

Bernoulli's equation

$$p_\infty + \frac{\rho}{2}U_\infty^2 = p + \frac{\rho}{2}q_\theta^2 \quad \longrightarrow \quad p - p_\infty = \frac{1}{2}\rho U_\infty^2 (1 - 4\sin^2\theta)$$

$$C_p = \frac{p - p_\infty}{(1/2)\rho U_\infty^2} = (1 - 4\sin^2\theta) \qquad (3.104)$$



3.11 Flow around a Cylinder (Continue)

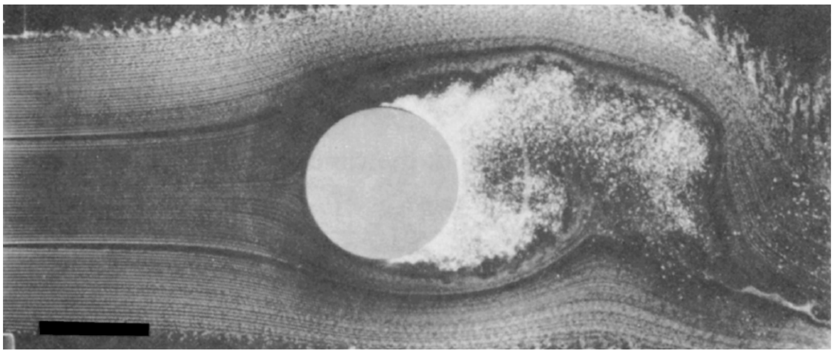
The fluid dynamic force acting on the cylinder

$$L = \int_0^{2\pi} -pR\,d\theta\sin\theta = \int_0^{2\pi} -(p - p_\infty)R\,d\theta\sin\theta$$

$$= \frac{-1}{2}\rho U_\infty^2 \int_0^{2\pi} (1 - 4\sin^2\theta)R\sin\theta\,d\theta = 0 \qquad (3.105)$$

$$D = \int_0^{2\pi} -pR\,d\theta\cos\theta = \int_0^{2\pi} -(p - p_\infty)R\,d\theta\cos\theta$$

$$= \frac{-1}{2}\rho U_\infty^2 \int_0^{2\pi} (1 - 4\sin^2\theta)R\cos\theta\,d\theta = 0 \qquad (3.106)$$



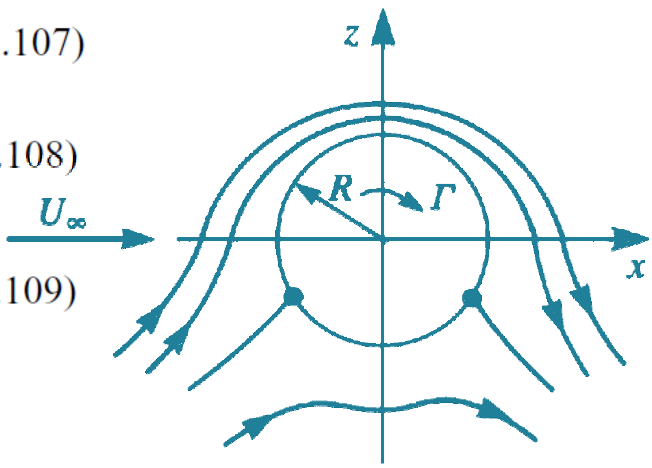
3.11 Flow around a Cylinder (Continue)

Lifting condition with a clockwise vortex with strength Γ situated at the origin

$$\Phi = U_\infty\cos\theta\left(r + \frac{R^2}{r}\right) - \frac{\Gamma}{2\pi}\theta \qquad (3.107)$$

$$q_r = U_\infty\cos\theta\left(1 - \frac{R^2}{r^2}\right) \qquad (3.108)$$

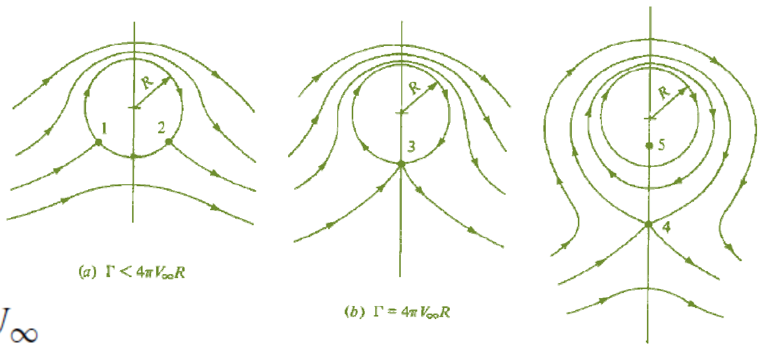
$$q_\theta = -U_\infty\sin\theta\left(1 + \frac{R^2}{r^2}\right) - \frac{\Gamma}{2\pi r} \qquad (3.109)$$



The stagnation points : at $r = R$ $q_\theta = 0$

$$q_\theta = -2U_\infty\sin\theta - \frac{\Gamma}{2\pi R}$$

$$\sin\theta_s = -\frac{\Gamma}{4\pi RU_\infty} \qquad (3.111)$$



(a) $\Gamma < 4\pi V_\infty R$

(b) $\Gamma = 4\pi V_\infty R$

(c) $\Gamma > 4\pi V_\infty R$

Stagnation points located at an angular position θ_s lie on the cylinder $\Gamma \le 4\pi RU_\infty$

3.11 Flow around a Cylinder (Continue)

The lift and drag will be found by using Bernoulli's equation

$$L = \int_0^{2\pi} -(p - p_\infty) R \, d\theta \sin\theta$$

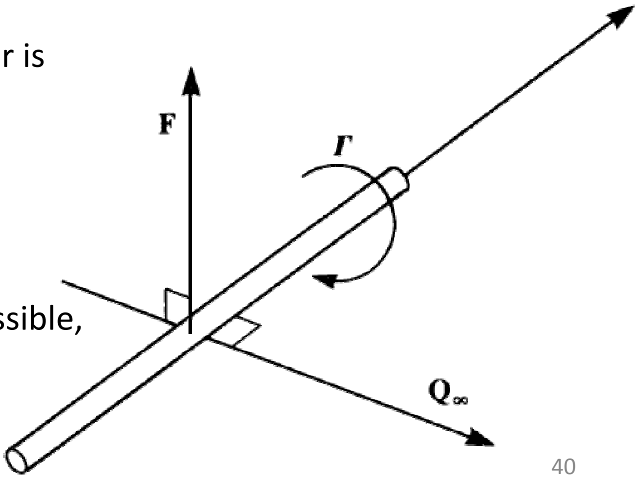
$$= -\int_0^{2\pi} \left[\frac{\rho U_\infty^2}{2} - \frac{\rho}{2} \left(2U_\infty \sin\theta + \frac{\Gamma}{2\pi R} \right)^2 \right] \sin\theta \, R \, d\theta$$

$$= \frac{\rho U_\infty \Gamma}{\pi} \int_0^{2\pi} \sin^2\theta \, d\theta = \rho U_\infty \Gamma \qquad (3.112)$$

Kutta–Joukowski theorem:

The lift per unit span on a lifting airfoil or cylinder is proportional to the circulation

$$\mathbf{F} = \rho \mathbf{Q}_\infty \times \mathbf{\Gamma} \qquad (3.113)$$



The resultant aerodynamic force in an incompressible, inviscid, irrotational flow
acts in a direction normal to the free stream

3.12 Flow around a Sphere

Superposition of a Three-Dimensional Doublet and Free Stream

The velocity potential:

$$\Phi = U_\infty r \cos\theta + \frac{\mu}{4\pi} \frac{\cos\theta}{r^2}$$

The velocity field:

$$q_r = \frac{\partial \Phi}{\partial r} = \left(U_\infty - \frac{\mu}{2\pi r^3} \right) \cos\theta$$

$$q_\theta = \frac{1}{r} \frac{\partial \Phi}{\partial \theta} = -\left(U_\infty + \frac{\mu}{4\pi r^3} \right) \sin\theta$$

$$q_\varphi = \frac{1}{r \sin\theta} \frac{\partial \Phi}{\partial \varphi} = 0$$

At the sphere surface, where *r = R,* the zero normal flow boundary condition, $q_r = 0$

$$q_r = \left(U_\infty - \frac{\mu}{2\pi R^3} \right) \cos\theta = 0 \quad \longrightarrow \quad \begin{cases} \theta = \pi/2, 3\pi/2 \\ \mu = U_\infty 2\pi R^3 \end{cases}$$

doublet strength

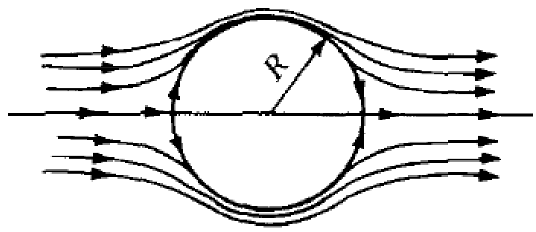
3.12 Flow around a Sphere

The flowfield around a sphere with a radius *R*

$$\Phi = U_\infty \cos\theta \left(r + \frac{R^3}{2r^2} \right)$$

$$q_r = U_\infty \cos\theta \left(1 - \frac{R^3}{r^3} \right)$$

$$q_\theta = -U_\infty \sin\theta \left(1 + \frac{R^3}{2r^3} \right)$$



The velocity components at $r = R$

$$q_r = 0, \qquad q_\theta = -\frac{3}{2} U_\infty \sin\theta$$

Stagnation Points $\begin{cases} \theta = 0 \\ \theta = \pi \end{cases}$

The pressure distribution is obtained now with Bernoulli's equation

$$p - p_\infty = \frac{1}{2}\rho U_\infty^2 \left(1 - \frac{9}{4}\sin^2\theta \right)$$

$$C_p = \frac{p - p_\infty}{(1/2)\rho U_\infty^2} = \left(1 - \frac{9}{4}\sin^2\theta \right)$$

3.12 Flow around a Sphere

Due to symmetry, lift and drag will be zero, as in the case of the flow over the cylinder.
However, the lift on a hemisphere is not zero
The lift force acting on the hemisphere's upper surface

$$L = -\int (p - p_\infty) \sin\theta \ \sin\varphi \, dS$$

surface element on the sphere

Particular interest in the field of road vehicle aerodynamics

$$dS = (R \sin\theta \, d\varphi)(R \, d\theta)$$

$$L = -\int_0^\pi \int_0^\pi \frac{1}{2}\rho U_\infty^2 \left(1 - \frac{9}{4}\sin^2\theta \right) R^2 \sin^2\theta \sin\varphi \, d\theta \, d\varphi$$

$$= -\frac{1}{2}\rho U_\infty^2 \int_0^\pi \left(1 - \frac{9}{4}\sin^2\theta \right) 2R^2 \sin^2\theta \, d\theta$$

$$= -\rho R^2 U_\infty^2 \left(\frac{\pi}{2} - \frac{27\pi}{32} \right) = \frac{11}{32}\pi\rho R^2 U_\infty^2$$

The lift and drag coefficients due to the upper surface

$$C_L \equiv \frac{L}{(1/2)\rho U_\infty^2 (\pi/2) R^2} = \frac{11}{8}$$

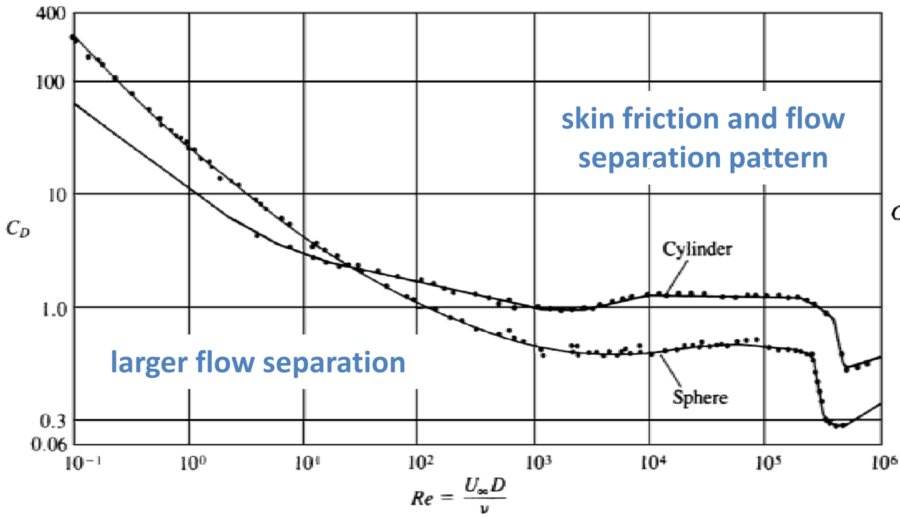
$$C_D \equiv \frac{D}{(1/2)\rho U_\infty^2 (\pi/2) R^2} = 0$$

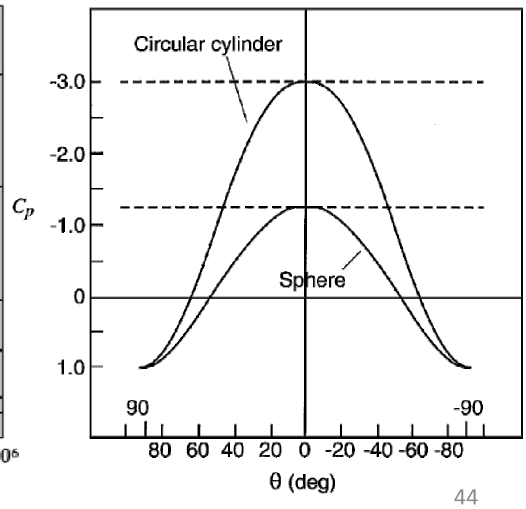
For the complete configuration the forces due to the pressure distribution on the flat, lower surface of the hemisphere must be included too, in this calculation.

3.13 The Flow over the Cylinder and the Sphere

For the 3D case the suction pressures are much smaller (relieving effect). Experimental data for the sphere show that the flow separates too but that the low pressure in the rear section is smaller. Consequently, the actual drag coefficient of a sphere is less than that of an equivalent cylinder

Separation & Viscous Friction → *d'Alembert's paradox*





3.14 Surface Distribution of the Basic Solutions

A solution to flow over arbitrary bodies can be obtained by distributing elementary singularity solutions over the modeled surfaces.

Investigating nature of each of the elementary solutions & type of discontinuity across the surface needs to be examined

For simplicity, the 2D point elements will be distributed continuously along the x axis in the region $x_1 \rightarrow x_2$.

Source Distribution

Source distribution of strength per length σ(x) along the x

The influence of this distribution at a point P(x, z)

$$\Phi(x,z) = \frac{1}{2\pi}\int_{x_1}^{x_2} \sigma(x_0) \ln\sqrt{(x-x_0)^2+z^2}\,dx_0 \qquad (3.130)$$

$$u(x,z) = \frac{1}{2\pi}\int_{x_1}^{x_2} \sigma(x_0)\frac{x-x_0}{(x-x_0)^2+z^2}\,dx_0 \qquad (3.131)$$

$$w(x,z) = \frac{1}{2\pi}\int_{x_1}^{x_2} \sigma(x_0)\frac{z}{(x-x_0)^2+z^2}\,dx_0 \qquad (3.132)$$

3.14 Surface Distribution of the Basic Solutions

As $z \to 0$ the integrand in Eq. (3.132) is zero except when $x_0 = x$
$\sigma(x_0)$ can be moved out of the integral and replaced by $\sigma(x)$

approaching $z = 0$ $\left\{ w^+, w^- \right.$ —Eq. (3.132)→ $w(x, 0+) = \lim_{z \to 0+} \frac{\sigma(x)}{2\pi} \int_{-\infty}^{\infty} \frac{z}{(x - x_0)^2 + z^2} dx_0$

introduce a new integration variable

$$\lambda = \frac{x - x_0}{z}$$

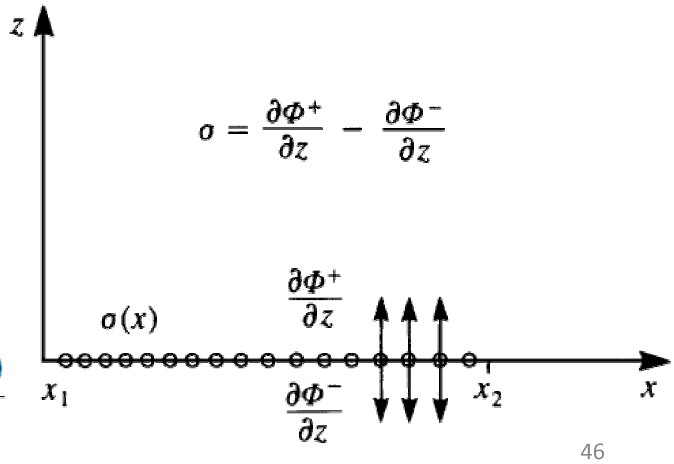
$$d\lambda = -\frac{dx_0}{z}$$

integration limits for $z \to 0^+$ become $\pm\infty$

$$w(x, 0+) = \lim_{z \to 0+} \frac{\sigma(x)}{2\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{1 + \lambda^2}$$

$$= \frac{\sigma(x)}{2\pi} \tan^{-1} \lambda |_{-\infty}^{\infty}$$

$$= \frac{\sigma(x)}{2\pi} \left[\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right] = \frac{\sigma(x)}{2}$$



3.14 Surface Distribution of the Basic Solutions

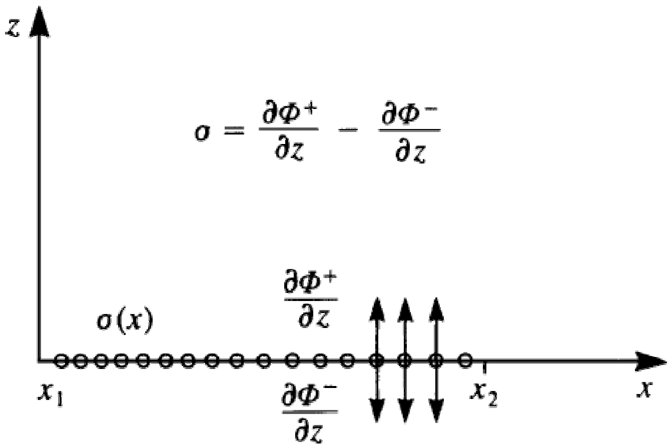
Therefore w(x, 0±) become

$$w(x, 0\pm) = \frac{\partial \Phi}{\partial z}(x, 0\pm) = \pm \frac{\sigma(x)}{2} \quad (3.135)$$

This element will be suitable to model flows that are symmetrical with respect to the x axis, and the total jump in the velocity component normal to the surface of the distribution is

$$w^+ - w^- = \sigma(x) \quad (3.136)$$

The u component is continuous across the x axis, and its evaluation needs additional considerations



3.14 Surface Distribution of the Basic Solutions

Doublet Distribution

a 2D doublet distribution, pointing in the z direction [**μ** = (0,μ)]

The influence of this distribution at a point P(x, z)

$$\Phi(x,z) = \frac{-1}{2\pi}\int_{x_1}^{x_2} \mu(x_0)\frac{z}{(x-x_0)^2+z^2}\,dx_0 \qquad (3.137)$$

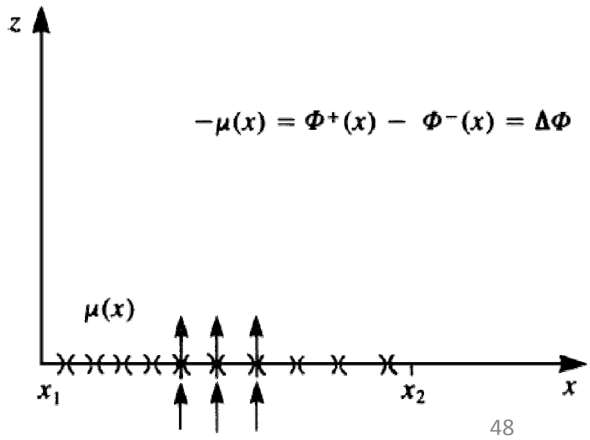
$$u(x,z) = \frac{1}{\pi}\int_{x_1}^{x_2} \mu(x_0)\frac{(x-x_0)z}{[(x-x_0)^2+z^2]^2}\,dx_0 \qquad (3.138)$$

$$w(x,z) = \frac{-1}{2\pi}\int_{x_1}^{x_2} \mu(x_0)\frac{(x-x_0)^2-z^2}{[(x-x_0)^2+z^2]^2}\,dx_0 \qquad (3.139)$$

Note: velocity potential in Eq. (3.137) is identical in form to w component of the source (Eq. (3.132))

Approaching the surface, at z= 0 ± element creates a jump in the velocity potential.

$$\Phi(x, 0\pm) = \mp\frac{\mu(x)}{2} \qquad (3.140)$$



3.14 Surface Distribution of the Basic Solutions

This leads to a discontinuous tangential velocity component

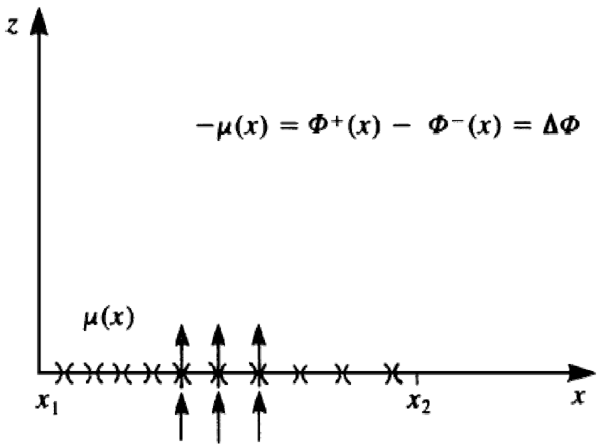
$$u(x, 0\pm) = \frac{\partial\Phi}{\partial x}(x, 0\pm) = \frac{\mp 1}{2}\frac{d\mu(x)}{dx} \qquad (3.141)$$

The circulation Γ(x) around a path surrounding the segment $x_1 \rightarrow x$ is

$$\Gamma(x) = \int_{x_1}^{x} u(x_0, 0+)\,dx_0 + \int_{x}^{x_1} u(x_0, 0-)\,dx_0 = -\mu(x) \qquad (3.142)$$

Which is equivalent to the jump in the potential

$$\Gamma(x) = \Phi(x, 0+) - \Phi(x, 0-) = -\mu(x) = \Delta\Phi(x) \qquad (3.143)$$



3.14 Surface Distribution of the Basic Solutions

Vortex Distribution

In a similar manner the influence of a vortex distribution at a point P(x, z) is

$$\Phi(x,z) = -\frac{1}{2\pi}\int_{x_1}^{x_2} \gamma(x_0)\tan^{-1}\frac{z}{x-x_0}\,dx_0 \qquad (3.144)$$

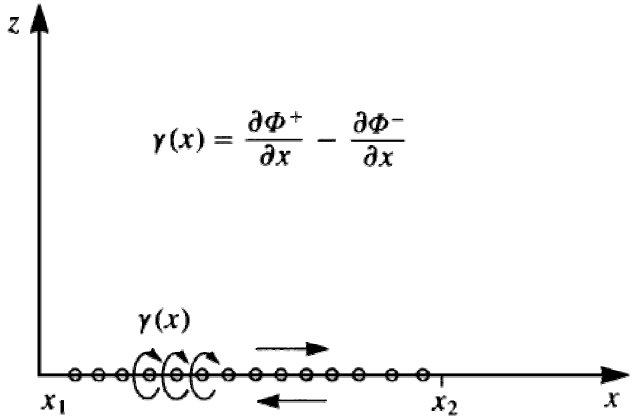
$$u(x,z) = \frac{1}{2\pi}\int_{x_1}^{x_2} \gamma(x_0)\frac{z}{(x-x_0)^2+z^2}\,dx_0 \qquad (3.145)$$

$$w(x,z) = -\frac{1}{2\pi}\int_{x_1}^{x_2} \gamma(x_0)\frac{x-x_0}{(x-x_0)^2+z^2}\,dx_0 \qquad (3.146)$$

The u component of the velocity is similar in form to Eqs. (3.132) and (3.137) and there is a jump in this component as z = 0±.

The tangential velocity component is

$$u(x,0\pm) = \frac{\partial\Phi}{\partial x}(x,0\pm) = \pm\frac{\gamma(x)}{2} \qquad (3.147)$$



3.14 Surface Distribution of the Basic Solutions

The contribution of this velocity jump to the potential jump, assuming that ϕ = 0 ahead of the vortex distribution, is

$$\Delta\Phi(x) = \Phi(x,0+) - \Phi(x,0-) = \int_{x_1}^{x}\frac{\gamma(x_0)}{2}\,dx_0 - \int_{x_1}^{x}\frac{-\gamma(x_0)}{2}\,dx_0$$

The circulation Γ is the closed integral of u(x, 0)dx,which is equivalent to that of Eq. (3.142).

$$\Gamma(x) = \Phi(x,0+) - \Phi(x,0-) = \Delta\Phi(x) \qquad (3.148)$$

Note: similar flow conditions can be modeled by either a vortex or a doublet distribution and the relation between these two distributions is

$$\Gamma = -\mu \qquad (3.149)$$

A comparison of Eq. (3.141) with Eq. (3.147) indicates that a vortex distribution can be replaced by an equivalent doublet distribution such that

$$\gamma(x) = -\frac{d\mu(x)}{dx} \qquad (3.150)$$